KURS
REKURENCJE, NOTACJA O, GRUPY i PIERŚCIENIE
(WYBRANE ZAGADNIENIA)

Lekcja 3
Wyznaczanie wzorów jawnych na ciąg rekurencyjny

Odpowiedzi do zadania domowego

Część 1: TEST

1) a
2) b
3) d
4) a
5) b
6) a
7) a
8) b
9) b
10) c

ODPOWIEDZI DO ZADAŃ

Zad. 1

a) $a_n = 5 - 2n$

b) $a_n = 2^n + 3^n$

c) $a_n = (-7)^n - \frac{8}{7} n \cdot (-7)^n$

d) $a_n = \frac{1}{5} \cdot 8^n - \frac{11}{5}(-2)^n$

e) $a_n = 2 \cdot 3^n$

Zad. 2

a) $a_{2^k} = 2^{k+2} - 3$

b) $a_{2^k} = -7k \cdot 2^{k-1} + 10 \cdot 2^{k-1} - 5$

c) $a_{2^k} = 2^k + 2^{k-1}\left(2^k - 1\right)$

Zad. 3

a) $a_n = 2\left(2^n - 1\right)$

Dowód indukcyjny (streszczenie):

KROK 1

$$a_1 = 2, \quad a_1 = 2\left(2^1 - 1\right) = 2 \cdot 1 = 2$$

$$a_2 = 2 \cdot 2 + 2 = 6, \quad a_2 = 2\left(2^2 - 1\right) = 2 \cdot 3 = 6$$

KROK 2

Założenie: $\begin{cases} a_1 = 2 \\ a_k = 2a_{k-1} + 2 \end{cases} \Leftrightarrow a_k = 2\left(2^k - 1\right), \quad dla\ 3 \le k \le n$

KROK 3

Teza: $\begin{cases} a_1 = 2 \\ a_{n+1} = 2a_n + 2 \end{cases} \Leftrightarrow a_{n+1} = 2\left(2^{n+1} - 1\right)$

$$a_{n+1} = 2a_n + 2 = 2\left(2\left(2^n - 1\right)\right) + 2 = 2\left(2^{n+1} - 2\right) + 2 = 2^{n+2} - 4 + 2 = 2^{n+2} - 2 = 2\left(2^{n+1} - 1\right)$$

CKD

b) $a_n = \frac{3}{2} + \frac{1}{2} \cdot (-1)^{n+1}$

Dowód indukcyjny (streszczenie):

KROK 1

$$a_1 = 2, \quad a_1 = \frac{3}{2} + \frac{1}{2} \cdot (-1)^{1+1} = \frac{3}{2} + \frac{1}{2} = 2$$

$$a_2 = 3 - 2 = 1, \quad a_2 = \frac{3}{2} + \frac{1}{2} \cdot (-1)^{2+1} = \frac{3}{2} - \frac{1}{2} = 1$$

KROK 2

Założenie: $\begin{cases} a_1 = 2 \\ a_k = 3 - a_{k-1} \end{cases} \Leftrightarrow a_k = \frac{3}{2} + \frac{1}{2} \cdot (-1)^{k+1}, \quad dla\ 3 \le k \le n$

KROK 3

Teza: $\begin{cases} a_1 = 2 \\ a_{n+1} = 3 - a_n \end{cases} \Leftrightarrow a_{n+1} = \frac{3}{2} + \frac{1}{2} \cdot (-1)^{n+2}$

$$a_{n+1} = 3 - a_n = 3 - \left(\frac{3}{2} + \frac{1}{2} \cdot (-1)^{n+1}\right) = 3 - \frac{3}{2} - \frac{1}{2} \cdot (-1)^{n+1} = \frac{3}{2} + (-1) \cdot \frac{1}{2} \cdot (-1)^{n+1} = \frac{3}{2} + \frac{1}{2} \cdot (-1)^{n+2}$$

c) $a_n = 2 - 3^{n-1}$

Dowód indukcyjny (streszczenie):

KROK 1

$$a_1 = 1, \quad a_1 = 2 - 3^{1-1} = 2 - 3^0 = 2 - 1 = 1$$

$$a_2 = 3 \cdot 1 - 4 = -1, \quad a_2 = 2 - 3^{2-1} = 2 - 3 = -1$$

KROK 2

Założenie: $\begin{cases} a_1 = 1 \\ a_k = 3a_{k-1} - 4 \end{cases} \Leftrightarrow a_k = 2 - 3^{k-1}, \quad dla\ 3 \le k \le n$

KROK 3

Teza: $\begin{cases} a_1 = 1 \\ a_{n+1} = 3a_n - 4 \end{cases} \Leftrightarrow a_{n+1} = 2 - 3^n$

$$a_{n+1} = 3a_n - 4 = 3\left(2 - 3^{n-1}\right) - 4 = 6 - 3^n - 4 = 2 - 3^n$$

d) $a_n = -\frac{19}{12} + \frac{17}{12} \cdot (-1)^{n+1}$

Dowód indukcyjny (streszczenie):

KROK 1

$$a_0 = -3, \quad a_0 = -\frac{19}{12} + \frac{17}{12} \cdot (-1)^{0+1} = -\frac{19}{12} - \frac{17}{12} = -\frac{36}{12} = -3$$

$$a_1 = \frac{1}{2 \cdot (-3)} = -\frac{1}{6}, \quad a_1 = -\frac{19}{12} + \frac{17}{12} \cdot (-1)^{1+1} = -\frac{19}{12} + \frac{17}{12} = -\frac{2}{12} = -\frac{1}{6}$$

KROK 2

Założenie: $\begin{cases} a_0 = -3 \\ a_k = \dfrac{1}{2a_{k-1}} \end{cases} \Leftrightarrow a_k = -\dfrac{19}{12} + \dfrac{17}{12} \cdot (-1)^{k+1}, \quad dla \quad 2 \le k \le n$

KROK 3

Teza: $\begin{cases} a_0 = -3 \\ a_{n+1} = \dfrac{1}{2a_n} \end{cases} \Leftrightarrow a_{n+1} = -\dfrac{19}{12} + \dfrac{17}{12} \cdot (-1)^{n+2}$

$$a_{n+1} = \frac{1}{2a_n} = \frac{1}{2\left(-\frac{19}{12} + \frac{17}{12} \cdot (-1)^{n+1}\right)} = \frac{1}{\frac{-19+17\cdot(-1)^{n+1}}{6}} = \frac{6}{-19+17\cdot(-1)^{n+1}} \cdot \frac{-19-17\cdot(-1)^{n+1}}{-19-17\cdot(-1)^{n+1}} =$$

$$\frac{6\left(-19-17\cdot(-1)^{n+1}\right)}{(-19)^2 - \left(17\cdot(-1)^{n+1}\right)^2} = \frac{6\left(-19-17\cdot(-1)^{n+1}\right)}{361-289} = \frac{6\left(-19-17\cdot(-1)^{n+1}\right)}{72} = \frac{-19-17\cdot(-1)^{n+1}}{12} =$$

$$= -\frac{19}{12} - \frac{17}{12} \cdot (-1)^{n+1} = -\frac{19}{12} - \frac{17}{12} \cdot \frac{1}{-1} \cdot (-1)^{n+2} = -\frac{19}{12} - \frac{17}{12} \cdot (-1) \cdot (-1)^{n+2} = -\frac{19}{12} + \frac{17}{12} \cdot (-1)^{n+2}$$

CKD.

Zad. 4

Dowód indukcyjny (streszczenie):

KROK 1

$$F_0 = 1 \le \left(\frac{1+\sqrt{5}}{2}\right)^0, \quad F_1 = 1 \le \left(\frac{1+\sqrt{5}}{2}\right)^1, \quad F_2 = 1+1 \le \left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$1 \le 1 \qquad 1 \le \frac{1+\sqrt{5}}{2} \quad / \cdot 2 \qquad 2 \le \frac{1+2\sqrt{5}+5}{4} \quad / \cdot 4$$

$$2 \le 1+\sqrt{5} \qquad 8 \le 6+2\sqrt{5}$$

$$1 \le \sqrt{5} \qquad 2 \le 2\sqrt{5}$$

KROK 2

Założenie: $F_k \le \left(\dfrac{1+\sqrt{5}}{2}\right)^k, \quad dla \; 3 \le k \le n$

KROK 3

Teza: $F_{n+1} \le \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

$$F_{n+1} = F_n + F_{n-1} \le \left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+\frac{2}{2}\right) =$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \frac{2\left(3+\sqrt{5}\right)}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \frac{6+2\sqrt{5}}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \frac{1+2\sqrt{5}+5}{4} =$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \frac{\left(1+\sqrt{5}\right)^2}{2^2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

Zad. 5

Dowód indukcyjny (streszczenie):

KROK 1

$$a_1 = 2 \ge \left(\frac{3}{2}\right)^1 \qquad a_2 = 3 \ge \left(\frac{3}{2}\right)^2 \qquad a_3 = 2+2\cdot 1 \ge \left(\frac{3}{2}\right)^3$$

$$2 \ge \frac{3}{2}, \qquad 3 \ge \frac{9}{4}, \quad 3 \ge 2\tfrac{1}{4}, \qquad 4 \ge \frac{27}{8}, \quad 4 \ge 3\tfrac{3}{8}$$

KROK 2

Założenie: $a_k \ge \left(\frac{3}{2}\right)^k, \quad dla\ 4 \le k \le n$

KROK 3

Teza: $a_{n+1} \ge \left(\frac{3}{2}\right)^{n+1}$

$$a_{n+1} = a_{n-1} + 2a_{n-2} \ge \left(\frac{3}{2}\right)^{n-1} + 2\left(\frac{3}{2}\right)^{n-2} = \left(\frac{3}{2}\right)^{n-2}\left(\frac{3}{2}+2\right) = \left(\frac{3}{2}\right)^{n-2} \cdot \frac{7}{2} = \left(\frac{3}{2}\right)^{n-2} \cdot \frac{28}{8} \ge \left(\frac{3}{2}\right)^{n-2} \cdot \frac{27}{8} =$$

$$= \left(\frac{3}{2}\right)^{n-2} \cdot \left(\frac{3}{2}\right)^{3} = \left(\frac{3}{2}\right)^{n+1}$$

Zad. 6

Dowód indukcyjny nierówności $\underset{n\geq 1}{\forall}\, 2^n \leq a_n \leq 2^{n+1}$ (streszczenie):

KROK 1

$$\begin{matrix} 2^1 \leq a_1 = 3 \leq 2^{1+1} \\ 2 \leq 3 \leq 4 \end{matrix}, \quad \begin{matrix} 2^2 \leq a_2 = 5 \leq 2^{2+1} \\ 4 \leq 5 \leq 8 \end{matrix}, \quad \begin{matrix} 2^3 \leq a_3 = 3\cdot 3 + 2\cdot 1 \leq 2^{3+1} \\ 8 \leq 11 \leq 16 \end{matrix}$$

KROK 2

Założenie: $\underset{k\geq 1}{\forall}\, 2^k \leq a_k \leq 2^{k+1}, \quad 4 \leq k \leq n$

KROK 3

Teza: $2^{n+1} \leq a_{n+1} \leq 2^{n+2}$

$$3\cdot 2^{n-1} + 2\cdot 2^{n-2} \leq a_{n+1} = 3a_{n-1} + 2a_{n-2} \leq 3\cdot 2^n + 2\cdot 2^{n-1}$$
$$3\cdot 2^{n-1} + 2^{n-1} \leq a_{n+1} = 3a_{n-1} + 2a_{n-2} \leq 3\cdot 2^n + 2^n$$
$$4\cdot 2^{n-1} \leq a_{n+1} = 3a_{n-1} + 2a_{n-2} \leq 4\cdot 2^n$$
$$2^2\cdot 2^{n-1} \leq a_{n+1} = 3a_{n-1} + 2a_{n-2} \leq 2^2\cdot 2^n$$
$$2^{n+1} \leq a_{n+1} = 3a_{n-1} + 2a_{n-2} \leq 2^{n+2}$$

Dowód indukcyjny zależności $a_n = 2a_{n-1} + (-1)^n$ (streszczenie):

KROK 1

$$a_1 = 3, \quad a_1 = 2\cdot 1 + (-1)^{1-1} = 3$$

$$a_2 = 5, \quad a_2 = 2\cdot 3 + (-1)^{2-1} = 5$$

$$a_3 = 3\cdot 3 + 2\cdot 1 = 11, \quad a_3 = 2\cdot 5 + (-1)^{3-1} = 11$$

KROK 2

Założenie: $a_n = 2a_{n-1} + (-1)^{n-1}, \quad dla\ 4 \leq k \leq n$

KROK 3

Teza: $a_{n+1} = 2a_n + (-1)^n$

$$P=2a_n+(-1)^n=2\left(2a_{n-1}+(-1)^{n-1}\right)+(-1)^n=4a_{n-1}+2\cdot(-1)^{n-1}+(-1)^n=4a_{n-1}+(-1)^{n-1}\left(2+(-1)\right)=$$
$$=4a_{n-1}+(-1)^{n-1}=3a_{n-1}+a_{n-1}+(-1)^{n-1}=3a_{n-1}+2a_{n-2}+(-1)^{n-2}+(-1)^{n-1}=3a_{n-1}+2a_{n-2}+(-1)^{n-1}\left(1+(-1)\right)=$$
$$=3a_{n-1}+2a_{n-2}=a_{n+1}=L$$

Zad. 7

$a_n=200\left(6-2^{-n}\right)$

$a_{10}=1199,80$ zł.

KONIEC